A linear programming approach to Fuglede's conjecture in $\mathbb{Z}_p^3$

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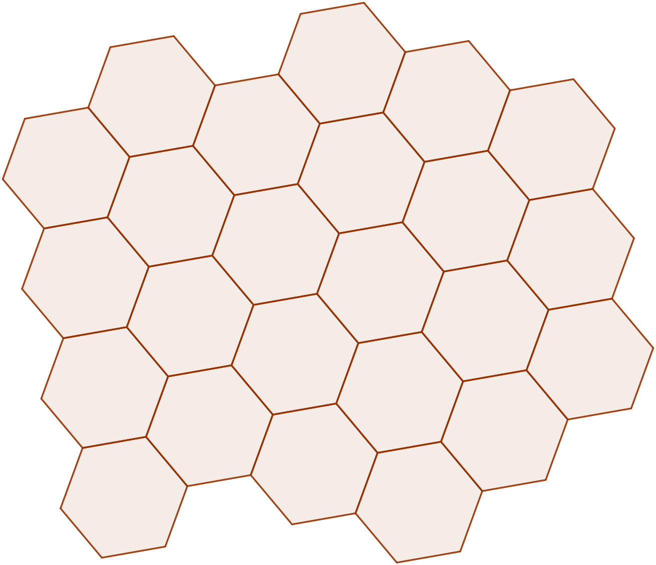
Tiling

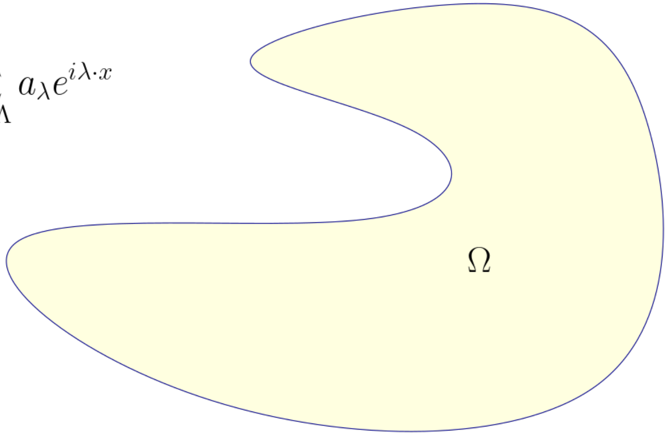


Spectrality

$$f \in L^2(\Omega)$$

$$\int_\Omega e^{i(\lambda-\lambda')\cdot x}\mathrm{d}x = 0, \quad \lambda \neq \lambda'$$

$$f(x) = \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda\cdot x}$$



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Fuglede's conjecture is still open for $d = 1, 2$. It is true for convex bodies (Lev, Matolcsi '22).

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A is spectral if there is a set of characters $B \subset \hat{G}$ that form an orthogonal basis on $L^2(A)$.

Results on Discrete Fuglede Conjecture (≤ 2 generators)

Cyclic groups $\mathbb{Z}_N$

1. If $N = p_1^m p_2^n p_3 \cdots p_k$, then $T \Rightarrow S$ (Łaba, Londner '22).
2. If $N = p_1^2 p_2^2 p_3^2 p_4 \cdots p_k$, then $T \Rightarrow S$ (Łaba, Londner '25).
3. If $N = p^m q^n$ and one of the following holds:
   - $p < q$ and $m \leq 9$ or $n \leq 6$,
   - $p^{m-2} < q^4$,

   then $S \Rightarrow T$ (M. '22).
4. If $N = pqrs$, then $S \Rightarrow T$ (Kiss, M, Somlai, Vizer '22).
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Two generators

1. If $G = \mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$, then $T \Rightarrow S$ and $S \Rightarrow T$ (Kiss, Somlai, Villano '23).
2. If $G = \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, then $T \Rightarrow S$ and $S \Rightarrow T$ (Zhang, '23)

Results on Discrete Fuglede Conjecture (≥ 3 generators)

Three generators

1. If $G = \mathbb{Z}_8^3$, then $S \not\Rightarrow T$ (Kolountzakis, Matolcsi '06).
2. If $G = \mathbb{Z}_n^3$, where $24 \mid n$ and n sufficiently large, then $T \not\Rightarrow S$ (Farkas, Matolcsi, Móra '06).
3. If $G = \mathbb{Z}_p^3$, then $T \Rightarrow S$ (Aten et al. '17).
4. If $G = \mathbb{Z}_p^3$ and $p \leq 7$, then $S \Rightarrow T$ (Fallon, Mayeli, Villano)

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≥ 4 generators

1. If $G = \mathbb{Z}_p^4$ and p odd, then $S \not\Rightarrow T$ (Ferguson, Sothanaphan '20).
2. If $G = \mathbb{Z}_2^{10}$, then $S \not\Rightarrow T$ (Ferguson, Sothanaphan '20).
3. If $G = \mathbb{Z}_2^6$, then $T \Rightarrow S$ and $S \Rightarrow T$ (Ferguson, Sothanaphan '20).

Results on Discrete Fuglede Conjecture (summary)

Fundamental Theorem on finite Abelian groups

If G is finite Abelian group, then

$$G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_k},$$

where $d_1 \mid d_2 \mid \cdots \mid d_k$.

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Summary of results

1. If G has at least 10 generators, then $S \not\Rightarrow T$.
2. If G has odd order and at least 4 generators, then $S \not\Rightarrow T$.
3. If G has at most 2 generators, we only have positive results so far.

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Fourier transform

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Discrete Fourier Analysis

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Inverse Fourier transform: $f(x) = \frac{1}{|G|} \sum_{\xi \in \hat{G}} \hat{f}(\xi)\xi(x)$.
Convolution: $f * g(x) = \sum_{y \in G} f(x-y)g(y)$. $\widehat{f * g} = \hat{f} \cdot \hat{g}$.
Parseval: $\mathbf{U} = \frac{1}{\sqrt{|G|}}\mathbf{F}$ is unitary: $|G| \sum_{x \in G} |f(x)|^2 = \sum_{\xi \in \hat{G}} |\hat{f}(\xi)|^2$.

Orthogonal characters

Restricting inner products on $A \subset G$:

$$\langle f, g \rangle_A = \sum_{x \in A} f(x)\overline{g(x)} = \langle f|_A, g|_A \rangle.$$

$\xi, \psi \in \hat{G}$ are orthogonal on A if $\langle \xi, \psi \rangle_A = 0$ (Notation: $\xi \perp \psi$.)

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$B \subset \hat{G}$ is a set of orthogonal characters of $A \subset G$, if for every $\xi \neq \psi$, $\xi, \psi \in B$ we have

$$0 = \langle \xi, \psi \rangle_A = \sum_{x \in A} (\xi\psi^{-1})(x) = \hat{\mathbf{1}}_A(\xi\psi^{-1})$$

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If in addition $|B| = |A|$, then B is a spectrum of A (it always holds $|B| \leq |A|$).

$G = \mathbb{Z}_p^3$

Fix an isomorphism $G \cong \hat{G}$, under the map $x \mapsto \xi_x$, where $\xi_x(y) = \zeta_p^{\langle x,y \rangle}$, with $\zeta_p = e^{2\pi i/p}$ and

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

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Zeros of $\widehat{\mathbf{1}}_A$

1. $\widehat{\mathbf{1}}_A(x) = 0 \Rightarrow \widehat{\mathbf{1}}_A(\lambda x) = 0$, $\forall \lambda \in \mathbb{Z}_p^*$, using the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. So, $Z(\widehat{\mathbf{1}}_A)$ is a union of *punctured lines*.
2. If $\widehat{\mathbf{1}}_A(x) = 0$, then A is equidistributed with respect to the p parallel planes of $x^{\perp}$. In particular, $p \mid |A|$.

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So, if we assume that A is nontrivial, so that $|A| = p$ or p^2, we get that both $\widehat{\mathbf{1}}_A$ and $\widehat{\mathbf{1}}_T$ must vanish somewhere.

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Finally, suppose that $|A| = p^2$, hence $|T| = p$. Consider a line L through two points of T; now let a plane H through the origin that is orthogonal to the direction of L. For any $x \in H^*$, we must have $\widehat{\mathbf{1}}_T(x) \neq 0$, since T is not equidistributed with respect to the planes parallel to $x^\perp$ (the one containing L has at least 2 elements of T). Therefore, (1) yields $H^* \subseteq Z(\widehat{\mathbf{1}}_A)$, and H is a spectrum of A, since $H - H = H \subseteq Z(\widehat{\mathbf{1}}_A) \cup \{0\}$ and $|H| = |A|$.

Delsarte's method

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Witness function

A function $h : G \to \mathbb{R}$ is called a *witness function* with respect to E if

(a) h is even and $h(x) \leq 0$, $\forall x \in G \setminus E$.

(b) $\hat{h} \geq 0$, $\hat{h}(0) > 0$.

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Theorem (Delsarte '72)

With B, E, h as above, it holds

$$|B| \leq |G| \cdot \frac{h(0)}{\hat{h}(0)}.$$

Spectrum

If there is a witness $h : G \to \mathbb{R}$ for $E = G \setminus Z(\hat{\mathbf{1}}_A)$ such that

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Remark

$h = \widehat{\mathbf{1}_A * \mathbf{1}_{-A}} = |\hat{\mathbf{1}}_A|^2$ *is a witness function for E which achieves equality, i. e.* $|G| \cdot h(0)/\hat{h}(0) = |A|$.

Balanced functions

Balanced (or ray-type) functions

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Balanced witness function

If h is a witness function for a union of lines E, then g is also a witness function for E, where

$$g(x) = \frac{1}{p-1} \sum_{\lambda \in \mathbb{Z}_p^*} h(\lambda x)$$

is in addition a balanced function.

Passing to $\mathbf{P}\mathbb{F}_p^2$

$[x : y : z] = [\lambda x : \lambda y : \lambda z]$ for $\lambda \neq 0$. The affine plane is included in $\mathbf{P}\mathbb{F}_p^2$ via the map $(x, y) \mapsto [x : y : 1]$; for $z = 0$ we get the line at infinity.

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If S is a union of punctured lines in $\mathbb{Z}_p^3$, then the corresponding set of points in $\mathbf{P}\mathbb{F}_p^2$ is denoted by $\tilde{S}$.

If L is a line through O, then:

$$\hat{\mathbf{1}}_O = \mathbf{1}_{\mathbb{Z}_p^3}, \quad \hat{\mathbf{1}}_L = p\mathbf{1}_{L^\perp}, \quad \hat{\mathbf{1}}_{L^*} = p\mathbf{1}_{L^\perp} - \mathbf{1}_{\mathbb{Z}_p^3}$$

Fourier analysis on the finite projective plane

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Functions on projective plane

For $f : \mathbb{Z}_p^3 \to \mathbb{C}$ balanced, define $\tilde{f} : \mathbf{P}\mathbb{F}_p^2 \cup \{O\} \to \mathbb{C}$ as $\tilde{f}([x : y : z]) = f(x, y, z)$, $\tilde{f}(O) = f(O)$. The Fourier transform is defined to satisfy $\hat{\tilde{f}} = \tilde{\hat{f}}$.

Fourier analysis on the finite projective plane

If L is a line through O, then:

$$\hat{\mathbf{1}}_O = \mathbf{1}_{\mathbb{Z}_p^3}, \quad \hat{\mathbf{1}}_L = p\mathbf{1}_{L^\perp}, \quad \hat{\mathbf{1}}_{L^*} = p\mathbf{1}_{L^\perp} - \mathbf{1}_{\mathbb{Z}_p^3}$$

Functions on projective plane

For $f : \mathbb{Z}_p^3 \to \mathbb{C}$ balanced, define $\tilde{f} : \mathbf{P}\mathbb{F}_p^2 \cup \{O\} \to \mathbb{C}$ as $\tilde{f}([x : y : z]) = f(x, y, z)$, $\tilde{f}(O) = f(O)$. The Fourier transform is defined to satisfy $\hat{\tilde{f}} = \tilde{\hat{f}}$.

Abusing notation, we write $O = [0 : 0 : 0]$. For $P = [x : y : z] \in \mathbf{P}\mathbb{F}_p^2$ define

$$P^\perp = \left\{ Q = [u : v : w] \in \mathbf{P}\mathbb{F}_p^2 : xu + yv + zw = 0 \right\}.$$

$$\hat{\delta}_P = p\delta_{P^\perp} + p\delta_O - \mathbf{1}, \quad \hat{\delta}_O = \mathbf{1}.$$

Blocking sets

Definition

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Facts:

- If Z is a blocking set, then so is Z^c.
- If $A \subset \mathbb{Z}_p^3$ is spectral, then $p \mid |A|$. If $|A| = p$ or p^2, then it tiles. If $|A| > p^2$, then $A = \mathbb{Z}_p^3$. Otherwise, $|A| = pk$, with $1 < k < p$ and

$$\widetilde{Z(\hat{\mathbf{1}}_A)} = \left\{[x : y : z] \in \mathbf{P}\mathbb{F}_p^2 : \hat{\mathbf{1}}_A(x,y,z) = 0\right\} = Z^c$$

  is a blocking set, and so is $Z = \widetilde{\operatorname{supp}\hat{\mathbf{1}}_A}$ (Fallon, Mayeli, Villano '19).
- Let Z' be the smallest blocking set such that $Z' \subset Z$. Then (Bruen, Thas '77),

$$|Z'| \leq p\sqrt{p} + 1.$$

Finding the witness function

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$$\tilde{h} = \delta_{Z'} + (|Z'| - p)\delta_O.$$

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h is a witness function for $E = \mathbb{Z}_p^3 \setminus Z(\hat{\mathbf{1}}_A)$:

- The first condition ($h \leq 0$ *outside* E) is satisfied, as $\mathrm{supp}\tilde{h} \subset Z \cup \{O\}$.
- The second condition (*positivity of* $\hat{h}$) is satisfied:

$$\hat{\tilde{h}} = p\left(\sum_{P\in Z'} \delta_{P^\perp} + |Z'|\delta_O - \mathbf{1}\right),$$

so that for $Q \in \mathbf{P}\mathbb{F}_p^2$

$$\hat{\tilde{h}}(Q) = p\left(\sum_{P\in Z'} \delta_{P^\perp}(Q) - 1\right) = p\left(\sum_{P\in Z'} \delta_{Q^\perp}(P) - 1\right) = p(|Z' \cap Q^\perp| - 1) \geq 0$$

and $\hat{\tilde{h}}(O) = p(|Z'| - 1) > 0$.

Finding the witness function

Suppose that $B \subset \hat{G}$ is a (maximal) set of pairwise orthogonal characters on A. Delsarte's method with witness function h gives us

$$|B| \leq |G| \cdot \frac{h(0)}{\hat{h}(0)} = p^3 \cdot \frac{\tilde{h}(O)}{\hat{\tilde{h}}(O)} = p^3 \cdot \frac{|Z'| - p}{p(|Z'| - 1)} = p^2 \left(1 - \frac{p-1}{|Z'| - 1}\right).$$

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$$k > p - \frac{p-1}{\sqrt{p}}.$$

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Theorem (M. '24)

If $A \subset \mathbb{Z}_p^3$ and

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Remark

- *It takes care about $\sqrt{p}$ multiples of p between $2p$ and $(p-1)p$.*
- *Previously known only for $k = p - 2$ or $p - 1$ (Fallon, Mayeli, Villano '19).*

Work in progress - Open questions

1. Could Z' be smaller? At any rate, not smaller than $\frac{3}{2}(p+1)$; in this case, if

$$p \cdot \frac{p^2+5p}{3p+1} < |A| < p^2,$$

then A is not spectral, using the same method

2. Could Z intersect every line in more than one points? Z either intersects every line at 3 points at least, or the points of A are distributed in k parallel planes, each having exactly p points of A.

3. If Z is a t-blocking set (i. e. it intersects every line at $\geq t$ points), then

$$\tilde{h} = \delta_{Z'} + (|Z'| - tp)\delta_O$$

is a witness function with respect to $E = \mathbb{Z}_p^3 \setminus Z(\hat{\mathbf{1}}_A)$, where Z' is a minimal t-blocking subset of Z. Applying Delsarte's method on h and using bounds on the size of minimal 3-blocking sets, yield that A is not spectral for $\approx \sqrt{3p}$ values of k.

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Thank you!